CARLEMAN'S FORMULA OF A SOLUTIONS OF THE POISSON EQUATION IN BOUNDED DOMAIN

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We suggest an explicit continuation formula for are a solution to the Cauchy problem for the Poisson equation in a domain from its values and the values of its normal derivative on part of the boundary. We construct an continuation formula of this problem based on the Carleman-Yarmuhamedov function method.

Keywords: ill-posed problem, regular solution, Carleman - Yarmuhamedov function, Carleman formula, Mittag-Leffler entire function.

Poisson equation or potential equation [1]

$$-\Delta U(x) \equiv -\sum_{i=1}^3 \frac{\partial^2 U}{\partial x_i^2} = f(x), \quad (1)$$

is the classical example for second order elliptic partial differential equations and it is a mathematical model to some important physical phenomena. It has applications in many different areas such as plasma physic, electrocardiography, and corrosion non-destructive evaluation (e.g., [2], [3],[4], [5], [6]).

In this paper, we offer an explicit formula for reconstruction of a solution of the Poisson equation in bounded domain from its values and the values of its normal derivative on part of the boundary, i.e., we give an explicit continuation formula for a solution to the Cauchy problem for the Poisson equation.

Let R^3 is the third dimensional real Euclidean space,

$$x = (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3) \in R^3, \quad x' = (x_1, x_2), \quad y' = (y_1, y_2) \in R^2,$$

$$s = \alpha^2 = |y' - x'| = (y_1 - x_1)^2 + (y_2 - x_2)^2, \quad r^2 = s + (y_3 - x_3)^2 = |y - x|^2, \quad \tau = tg \frac{\pi}{2\rho},$$

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$\rho > 1$, $G_\rho = \{y : |y'| < \tau y_3, y_3 > 0\}$, $\partial G_\rho = \{y : |y'| = \tau y_3, y_3 > 0\}$,
 $\overline{G}_\rho = G_\rho \cup \partial G_\rho$, $\varepsilon, \varepsilon_1$ and ε_2 sufficiently small positive constants,
 Ω_ρ — is a bounded simply connected domain whose boundary in R^3 whose
boundary $\partial\Omega_\rho$ consists of a part of the conic surface $T \equiv \partial G_\rho$ and a smooth
surface S , lying inside the cone $\overline{G}_\rho$. Function f Hölder continuous with
exponent $\lambda \in (0, 1)$, i.e., $f \in C^{s,\lambda}(\overline{\Omega}_\rho)$ and $s \in Z_+$.

Problem. Let we know the Cauchy data for a solution to system (0.1)
on the surface S :

$$U(y) = f_1(y), \frac{\partial U(y)}{\partial n} = f_2(y), y \in S \quad (2)$$

where $n = (n_1, n_2, n_3)$ is the unit outward-pointing normal to the surface
 $\partial\Omega$ at a point y , and f_1, f_2 are continuous vector-functions. Given $f_1(y)$ and
 $f_2(y)$ on S , find $U(x)$ $x \in \Omega$.

The Cauchy problem (2) for the Poisson equation (1) is well-known to be
ill-posed [7], [8]. Hadamard [9] noted that solution to problem is not stable.
Possibility of introducing a positive parameter σ , depending on the accuracy
of the initial data, was noticed by M. M. Lavrent'ev [10]. Uniqueness of the
solution follows from the general theorem by Holmgren [11]. Traditionally,
regularization techniques, such as Tikhonov regularization [12].

We suppose that a solution to the problem exists (in this event it is
unique) and continuously differentiable in the closed domain and the Cauchy
data are given exactly. In this case we establish an explicit continuation
formula. This formula enables us to state a simple and convenient criterion
for solvability of the Cauchy problem.

The result established here is a multidimensional analog of the theorems
and Carleman-type formulas [13], by G.M.Goluzin, V.I.Krylov, V.A.Fok, and
F.M.Kuni in the theory of holomorphic functions of one variable [14],[15].

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