INTEGRATION OF THE DISCRETE SINE-GORDON EQUATION WITH A SELF-CONSISTENT SOURCE

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This work is devoted to the application of the inverse scattering theory for integration of the discrete Sine-Gordon equation with a self-consistent source.

Keywords: discrete sine-Gordon equation, self-consistent source, discrete Dirac-type operator, scattering data, inverse scattering method

Hirota showed the integrability of discrete version of the sine-Gordon equation and found its Lax pair, Backlund transformations and N-soliton solutions [1]. In [2] generalization of Hirota's discretization scheme for the sine-Gordon equation was considered. The soliton solutions are obtained by extending the generalized inverse method [3] and the related linear spectral problem for the discrete sine-Gordon equation was studied in [4].

It needs to point out the sG equations and its close allies are valued in the investigation of a great variety of diverse fields[5], such as the study of surfaces with constant negative curvature, or integrable surfaces [6], elementary particle physics, quantum optics, Josephson junctions [7], nonlinear excitations in condensed matter physics[8], vortex structures in fluids and plasmas [9].

This study investigated the integration of the discrete sine-Gordon(sG) equation with a self-consistent source via the inverse scattering method.

The first investigation of the soliton equations with self-consistent sources has been considered in [10] and still attracts considerable attention in recent years [11-13, see also their reference].

We consider the following system of equations

$$\dot{\theta}_{n+1} - \dot{\theta}_n = 2(\sin \theta_{n+1} + \sin \theta_n) + \sum_{k=1}^N (f_{1,n+1}^k f_{1,n}^k + f_{2,n+1}^k f_{2,n}^k), n \in Z, \quad (1)$$

$$\theta_n(t)|_{t=0} = \theta_n^0, n \in Z, \quad (2)$$

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$$L_n(z_k, t) f_n^k = f_{n+1}^k, n \in Z \quad (3)$$

where $L_n(z, t) = zP_n(t) + \frac{1}{z}Q_n(t)$, $P_n(t) = \frac{1}{2} \begin{pmatrix} 1 + \cos \theta_n(t) & \sin \theta_n(t) \\ \sin \theta_n(t) & 1 - \cos \theta_n(t) \end{pmatrix}$,

$$Q_n(t) = \frac{1}{2} \begin{pmatrix} 1 - \cos \theta_n(t) & -\sin \theta_n(t) \\ -\sin \theta_n(t) & 1 + \cos \theta_n(t) \end{pmatrix},$$

$\theta_n = \theta_n(t)$ and $f_n^k = f_n(z_k, t)$, $\hat{f}_n^k = \sigma_2 f_n(z_k, t)$ column-vector functions satisfy the following normalizing conditions

$$\beta_k(t) = \sum_{i=-\infty}^{\infty} (f_i^k)^T (Q_i - P_i) \sigma_2 f_i^k, \quad \hat{\beta}_k(t) = \sum_{i=-\infty}^{\infty} (\hat{f}_i^k)^T (z_k^2 Q_i - P_i) \sigma_2 \hat{f}_i^k.$$

Here $\beta_k(t)$, $\hat{\beta}_k(t)$ are given scalar continuous functions and $\sigma_i (i = 1, 2, 3)$ are the usual Pauli matrices of rank 2.

The purpose of the work is to find the set of functions $\{\theta_n(t), f_n^k(t)\}, n \in Z$ supposing the existence in the class of functions

$$\lim_{|n| \rightarrow \pm\infty} \omega_n = \lim_{|n| \rightarrow \pm\infty} \sum_{j=n}^{\infty} \sin \theta_j(t) = 0, \quad \theta_n(t) = 0 \pmod{2\pi} \quad (4)$$

which is the solution of the considering (1)-(3) problem.

Theorem 1. If the set of functions $\{\theta_n(t), f_n^k(t)\}$ represent the solution of the (1)-(3) in the class of functions (4), then the scattering data of the $L_n(t)$ operator with the potential $\theta_n(t)$ satisfy the following time evolution equations

$$\begin{aligned} \dot{R}(z) &= -2 \left(\frac{z^2 + 1}{z^2 - 1} \right) R(z), \quad |z| = 1, z \neq \pm 1, \\ \dot{z}_k &= 0, \quad \dot{C}_k = \left(-2 \frac{z_k^2 + 1}{z_k^2 - 1} + \beta_k + \hat{\beta}_k \right) C_k, k = 1, \dots, N. \end{aligned}$$

The obtained results completely define the time evolution of the scattering data, which allows us to solve the problem (1)-(3) by using the method of the inverse spectral problem of (3) [see 4].

Литература

1. Hirota R. 1977, J. Phys. Soc. Japan, 43, 2079.
2. Orfanidis S. 1978 J. Phys. Rev. D 18 3822.
3. Levi D., Ragnisco O., Bruschi M. 1980, Nuovo Cimento A 58 56.
4. Piliñi L. and Levi D. The Inverse Scattering Transform for Solving the Discrete Sine-Gordon Equation // Physics Letters A, 92 (1982), 1, pp. 5-81982.

5. *Fritz Gesztesy, Helge Holden*. A Local Sine-Gordon Hierarchy and its Algebro-Geometric Solutions // arXiv:solv-int/9707010
6. *Bobenko A. I.* Constant mean curvature surfaces and integrable systems // Russ. Math. Surv.46:4 , 1991, 1–45.
7. *Borisov A.B., Kiseliev V. V.* Topological defects in incommensurate magnetic and crystal structures and quasi-periodic solutions of the elliptic sine-Gordon equation // Physica D 31 ,1988, 49–64.
8. *Ting A.C., Chen H.H., Lee Y.C.* Exact solutions of a nonlinear boundary value problem: The vortices of the two-dimensional sinh-Poisson equation // Physica 26D , 1987, pp.37–66.
9. *Mel'nikov V.K.* Integration method of the Korteweg-de Vries equation with a self-consistent source // Phys.Lett. A, 133, 1988, pp.493-496.
10. *Da-Jun Zhang , Deng-guan Chen*. The N-Soliton Solutions of the sine-Gordon Equation with Self-Consistent Sources // Physica A 321, 2003, pp. 467–481.
11. *Khasanov A.B., Urazboev G.U.* On the sine-Gordon equation with a self-consistent source // Mat. Tr., 2008, Volume 11, Number 1, 153–166,
12. *Babajanov B.A., Fechkan M., Urazbaev G.U.* On the periodic Toda Lattice with self-consistent source // Communications in Nonlinear Science and Numerical Simulation. 2015; 22: 379-388.